

# Backward Uniqueness for a PDE Fluid-Structure Interaction

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## Abstract

In this work, we establish the so-called backward uniqueness property for a coupled system of partial differential equations (PDEs) which governs a certain fluid-structure interaction. In particular, a three-dimensional Stokes flow interacts across a boundary interface with a two-dimensional mechanical plate equation. By way of attaining this result, a certain estimate is obtained for the associated semigroup generator resolvent.

## 1 Statement of the Problem and Main Result

We consider here the problem of establishing the so-called backward uniqueness property for the partial differential equation (PDE) model given in [7] and [2], which describes a certain fluid-structure interactive dynamics. One novelty of this PDE system is the unique way in which the geometry affects the coupling between the fluid and the plate. Since the coupling involves the pressure term, the system cannot be solved via the classic Leray projector. Instead in [2] wellposedness is given via a semigroup formulation and proved via the Babuška-Brezzi theorem.

As explained in more detail in [3], the backward uniqueness property – described below in Theorem 3 – has important implications for the controllability of the system in the sense of PDE control theory.

As mentioned above, the geometry plays a particular role in the wellposedness of the system. The fluid chamber  $\mathcal{O} \subset \mathbb{R}^3$  will be a bounded domain with sufficiently smooth boundary. Moreover,  $\partial\mathcal{O} = \bar{\Omega} \cup \bar{S}$ , with  $\Omega \cap S = \emptyset$ . More specifically

$$\Omega \subset \{x = (x_1, x_2, 0)\}, \text{ and surface } S \subset \{x = (x_1, x_2, x_3) : x_3 \leq 0\}.$$

In consequence, if  $\nu(x)$  denotes the exterior unit normal vector to  $\partial\mathcal{O}$ , then

$$\nu|_{\Omega} = [0, 0, 1] \tag{1}$$

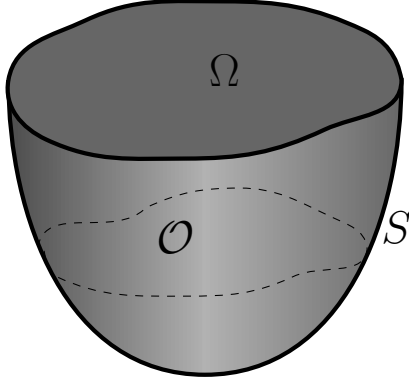


Figure 1: The Fluid-Structure Geometry

In addition,  $[\mathcal{O}, \Omega]$  is assumed to fall within one of the following classes:

- (G.1)  $\mathcal{O}$  is a convex domain with wedge angles  $< \frac{2\pi}{3}$ . Moreover,  $\Omega$  has smooth boundary, and  $S$  is a piecewise smooth surface;
- (G.2)  $\mathcal{O}$  is a convex polyhedron having angles  $< \frac{2\pi}{3}$ , and so then  $\Omega$  is a convex polyhedron with angles  $< \frac{2\pi}{3}$ .

The PDE model is as follows, with “rotational inertia parameter”  $\rho \geq 0$ , and in solution variables  $[w(x, t), w_t(x, t)]$ ,  $u(x, t) = [u^1(x, t), u^2(x, t), u^3(x, t)]$ , and  $p(x, t)$ :

$$w_{tt} - \rho \Delta w_{tt} + \Delta^2 w = p|_{\Omega} \quad \text{in } \Omega \times (0, T), \quad (2)$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega; \quad (3)$$

$$u_t - \Delta u + \nabla p = 0 \quad \text{in } \mathcal{O} \times (0, T), \quad (4)$$

$$\operatorname{div}(u) = 0 \quad \text{in } \mathcal{O} \times (0, T), \quad (5)$$

$$u = 0 \text{ on } S \quad \text{and } u = [u^1, u^2, u^3] = [0, 0, w_t] \quad \text{on } \Omega, \quad (6)$$

with initial conditions

$$[w(0), w_t(0), u(0)] = [w_1^*, w_2^*, u_0^*] \in \mathbf{H}_{\rho}. \quad (7)$$

Here, the finite energy space  $\mathbf{H}_{\rho}$  is given by

$$\begin{aligned} \mathbf{H}_{\rho} = & \left\{ [\omega_1, \omega_2, f] \in \left[ H_0^2(\Omega) \cap \widehat{L}^2(\Omega) \right] \times W_{\rho} \times \mathcal{H}_{\text{fluid}} \right. \\ & \left. \text{with } f \cdot \nu|_{\Omega} = [0, 0, f^3] \cdot [0, 0, 1] = \omega_2 \right\}, \end{aligned} \quad (8)$$

where

$$\widehat{L}^2(\Omega) = \left\{ \varpi \in L^2(\Omega) : \int_{\Omega} \varpi d\Omega = 0 \right\}; \quad (9)$$

$$\mathcal{H}_{\text{fluid}} = \{ f \in \mathbf{L}^2(\mathcal{O}) : \operatorname{div}(f) = 0; \ f \cdot \nu|_S = 0 \}; \quad (10)$$

and

$$W_{\rho} = \begin{cases} \widehat{L}^2(\Omega), & \text{if } \rho = 0, \\ H_0^1(\Omega) \cap \widehat{L}^2(\Omega), & \text{if } \rho > 0. \end{cases} \quad (11)$$

This Hilbert space  $\mathbf{H}_\rho$  of finite energy is endowed here with the following norm-inducing inner product:

$$\left( [\omega_1, \omega_2, f], [\tilde{\omega}_1, \tilde{\omega}_2, \tilde{f}] \right)_{\mathbf{H}_\rho} = (\Delta\omega_1, \Delta\tilde{\omega}_1)_\Omega + (\omega_2, \tilde{\omega}_2)_\Omega + \rho(\nabla\omega_2, \nabla\tilde{\omega}_2)_\Omega + (f, \tilde{f})_\mathcal{O}. \quad (12)$$

where  $(\cdot, \cdot)_\Omega$  and  $(\cdot, \cdot)_\mathcal{O}$  are the  $L^2$ -inner products on their respective geometries.

For the PDE system (2)-(7), semigroup well-posedness result was established in [2]; a proof of wellposedness, for  $\rho > 0$ , via a Galerkin method was also given in [7], which paper was primarily concerned with longtime behaviour of corresponding solutions, under nonlinear effects. We will presently give an explicit description of the modeling semigroup generator  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$ ; its construction in [2] hinged upon an appropriate elimination of the pressure variable  $p$  in (2)-(7). (As the no-slip boundary condition is necessarily not in play for the fluid variable  $u$ , one cannot merely invoke the classic Leray projector to eliminate the pressure term, as one would in uncoupled fluid flow; see e.g., [17].)

**Theorem 1** (See [2].) *The PDE model (2)-(7) is associated with a  $C_0$ -contraction semigroup  $\{e^{\mathcal{A}_\rho t}\}_{t \geq 0} \subset \mathcal{L}(\mathbf{H}_\rho)$ , the generator of which is given below in (24)-(26). Therewith, for any initial data  $[w_1^*, w_2^*, u_0^*] \in \mathbf{H}_\rho$ , the solution  $[w, w_t, u] \in C([0, T; \mathbf{H}_\rho])$  is given by*

$$\begin{bmatrix} w(t) \\ w_t(t) \\ u(t) \end{bmatrix} = e^{\mathcal{A}_\rho t} \begin{bmatrix} w_1^* \\ w_2^* \\ u^* \end{bmatrix}.$$

The main result of this paper – Theorem 3 below – deals with establishing the aforesaid backward uniqueness property for the contraction  $C_0$ -semigroup associated with the PDE model (2)-(7). The driving agent of our proof of Theorem 3 is the following abstract resolvent criterion for backward uniqueness.

**Theorem 2** (See [13], Theorem 3.1, p. 225.) *Let  $A$  be the infinitesimal generator of a s.c. semigroup  $e^{At}$  in a Banach space  $X$ . Assume that there exist constants  $a \in (\pi/2, \pi)$ ,  $r_0 > 0$ , and  $C > 0$ , such that*

$$\|\mathcal{R}(re^{\pm ia}; A)\|_{\mathcal{L}(X)} = \|(re^{\pm ia}I - A)^{-1}\|_{\mathcal{L}(X)} \leq C,$$

*for all  $r \geq r_0$ . Then the backward uniqueness property holds true; that is,  $e^{AT}x_0 = 0$  for  $T > 0$ ,  $x_0 \in X$ , implies  $x_0 = 0$ .*

By way of applying the abstract Theorem 2 to the modeling generator  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$  of (2)-(7), given explicitly in (24)-(26) below, we will consider the following resolvent relation with complex parameter  $\lambda = \alpha + i\beta$ , which is formally a “frequency domain” version of (2)-(7):

$$(\lambda I - \mathcal{A}_\rho) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} = \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \in \mathbf{H}_\rho. \quad (13)$$

Here, the pre-image  $[\omega_1, \omega_2, \mu] \in D(\mathcal{A}_\rho)$  and forcing term  $[\omega_1^*, \omega_2^*, \mu^*] \in \mathbf{H}_\rho$ .

With respect to the frequency domain parameter, we will furthermore impose that  $\lambda = \alpha + i\beta$  should obey the following criteria:

Criterion 1:  $\lambda = \alpha + i\beta = |\lambda|e^{\pm i\vartheta}$ , for fixed  $\vartheta \in (\frac{3\pi}{4}, \pi)$ . (And so on either of these two rays, we have  $0 < |\tan \vartheta| < 1$ ,  $|\beta| = |\alpha||\tan \vartheta|$ ,  $|\lambda|^2 = \alpha^2 + \beta^2 = \alpha^2(1 + \tan^2 \vartheta)$ .)

Criterion 2:  $|\alpha| > 0$  is sufficiently large.

Our main result can now be stated as follows:

**Theorem 3** (i) *With respect to the resolvent relation (13), or the equivalent fluid-structure PDE (29)-(30) below, let the Criteria 1 and 2 be in force. Then for all  $\rho \geq 0$ , the solution  $[\omega_1, \omega_2, \mu] \in D(\mathcal{A}_\rho)$  obeys the following bound, which is uniform for all  $\lambda = \alpha + i\beta = |\lambda| e^{\pm i\vartheta}$ , with fixed  $\vartheta \in (\frac{3\pi}{4}, \pi)$ , and  $|\alpha| > 0$  sufficiently large:*

$$\left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_\rho} \leq C_\vartheta \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_\rho}. \quad (14)$$

(ii) *In consequence, this estimate and Theorem 2 yields the conclusion that the fluid-structure  $C_0$ -contraction semigroup  $\{e^{\mathcal{A}_\rho t}\}_{t \geq 0}$  satisfies the backward uniqueness property: Namely, if for given  $T > 0$  and  $[w_1^*, w_2^*, u_0^*] \in \mathbf{H}_\rho$ , one has*

$$e^{\mathcal{A}_\rho T} \begin{bmatrix} w_1^* \\ w_2^* \\ u_0^* \end{bmatrix} = \vec{0}, \text{ then necessarily } \begin{bmatrix} w_1^* \\ w_2^* \\ u_0^* \end{bmatrix} = \vec{0}.$$

**Remark 4** *Unlike the coupled PDE examples in [13], [3], [4], the frequency domain estimate (14), for the fluid-structure solution  $[\omega_1, \omega_2, \mu]$  of (13), does not manifest a uniform rate of decay with respect to  $\text{Re}\lambda = \alpha$ . We are not certain that such a decay is actually possible. However by Theorem 2, the uniform bound (14) suffices to established the aforesaid backward uniqueness property.*

**Remark 5** *In the course of proof, the reader could infer that for the rotational inertial case  $\rho > 0$ , one will in fact have the uniform estimate (14) for any rays along the angle  $\vartheta \in (\frac{\pi}{2}, \pi)$ ,  $|\lambda|$  large enough.*

## 2 The Description of the Fluid-Structure Generator

Under the geometric conditions (G.1) and (G.2), we now tersely define the modeling generator  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$  which describes the dynamics (2)-(7), and for which Theorem 3 applies. Full details are given in [2] and [1].

To start, let  $A_D : L^2(\Omega) \rightarrow L^2(\Omega)$  be given by

$$A_D g = -\Delta g, \quad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega). \quad (15)$$

If we subsequently make the denotation for all  $\rho \geq 0$ ,

$$P_\rho = I + \rho A_D, \quad D(P_\rho) = \begin{cases} L^2(\Omega), & \text{if } \rho = 0, \\ D(A_D), & \text{if } \rho > 0, \end{cases} \quad (16)$$

then the mechanical PDE component (2)-(3) can be written as

$$P_\rho w_{tt} + \Delta^2 w = p|_\Omega \text{ on } (0, T).$$

Using the characterization from [9] that

$$D(P_\rho^{\frac{1}{2}}) = \begin{cases} L^2(\Omega), & \text{if } \rho = 0, \\ H_0^1(\Omega), & \text{if } \rho > 0, \end{cases}$$

then from (12) we can rewrite

$$\left([\omega_1, \omega_2, f], [\tilde{\omega}_1, \tilde{\omega}_2, \tilde{f}]\right)_{\mathbf{H}_\rho} = (\Delta\omega_1, \Delta\tilde{\omega}_1)_\Omega + (P_\rho^{\frac{1}{2}}\omega_2, P_\rho^{\frac{1}{2}}\tilde{\omega}_2)_\Omega + (f, \tilde{f})_{\mathcal{O}}. \quad (17)$$

Moreover, in order to eliminate the pressure – see [2] – we require the following “Robin” maps  $R_\rho$  and  $\tilde{R}_\rho$ :

$$R_\rho g = f \Leftrightarrow \left\{ \Delta f = 0 \text{ in } \mathcal{O}; \frac{\partial f}{\partial \nu} + P_\rho^{-1}f = g \text{ on } \Omega; \frac{\partial f}{\partial \nu} = 0 \text{ on } S \right\}. \quad (18)$$

$$\tilde{R}_\rho g = f \Leftrightarrow \left\{ \Delta f = 0 \text{ in } \mathcal{O}; \frac{\partial f}{\partial \nu} + P_\rho^{-1}f = 0 \text{ on } \Omega; \frac{\partial f}{\partial \nu} = g \text{ on } S \right\}. \quad (19)$$

By Lax-Milgram we then have

$$R_\rho \in \mathcal{L}(H^{-\frac{1}{2}}(\Omega), H^1(\mathcal{O})); \quad \tilde{R}_\rho \in \mathcal{L}(H^{-\frac{1}{2}}(S), H^1(\mathcal{O})). \quad (20)$$

(We are also using implicitly the fact that  $P_\rho^{-1}$  is positive definite, self-adjoint on  $\Omega$ .)

Therewith, it is shown in [2] that the pressure variable  $p(t)$  can be written pointwise in time as

$$p(t) = G_{\rho,1}(w(t)) + G_{\rho,2}(u(t)), \quad (21)$$

where

$$G_{\rho,1}(w) = R_\rho(P_\rho^{-1}\Delta^2 w); \quad (22)$$

$$G_{\rho,2}(u) = R_\rho(\Delta u^3|_\Omega) + \tilde{R}_\rho(\Delta u \cdot \nu|_S). \quad (23)$$

With these operators, we defined in [2] the generator  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$ , which is associated with the fluid structure system (2)-(7):

$$\mathcal{A}_\rho \equiv \begin{bmatrix} 0 & I & 0 \\ -P_\rho^{-1}\Delta^2 + P_\rho^{-1}G_{\rho,1}|_\Omega & 0 & P_\rho^{-1}G_{\rho,2}|_\Omega \\ -\nabla G_{\rho,1} & 0 & \Delta - \nabla G_{\rho,2} \end{bmatrix}; \quad (24)$$

with  $D(\mathcal{A}_\rho) = \{[w_1, w_2, u] \in \mathbf{H}_\rho \text{ satisfying :}$

$$(a) \ w_1 \in \mathcal{S}_\rho \equiv \begin{cases} H^4(\Omega) \cap H_0^2(\Omega), & \text{if } \rho = 0; \\ H^3(\Omega) \cap H_0^2(\Omega), & \text{if } \rho > 0; \end{cases} \quad (25)$$

$$(b) \ w_2 \in H_0^2(\Omega), \ u \in \mathbf{H}^2(\mathcal{O});$$

$$(c) \ u = \vec{0} \text{ on } S \text{ and } u = [0, 0, w_2] \text{ on } \Omega\}. \quad (26)$$

**Remark 6** Given data  $[w_1, w_2, u] \in D(\mathcal{A}_\rho)$ , note that as  $\Delta u \in L^2(\mathcal{O})$  and  $\text{div}(\Delta u) = 0$ , then by Theorem 1.2, p. 9 in [17], we have the trace regularity

$$\Delta u \cdot \nu|_{\partial\mathcal{O}} \in H^{-\frac{1}{2}}(\partial\mathcal{O}); \quad (27)$$

and so the pressure term

$$p \equiv G_{\rho,1}(w_1) + G_{\rho,2}(u) \in H^1(\mathcal{O}). \quad (28)$$

Thus,  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$  is indeed well-defined (see in particular the 2 – 3 and 3 – 3 entries of matrix  $\mathcal{A}_\rho$ ).

It is shown in [2] that  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$  is maximal dissipative, thereby giving rise to Theorem 1 above. The next Section is devoted to the proof of the main, backward uniqueness result.

### 3 Proof of Theorem 3

With  $\lambda = \alpha + i\beta$ , and with the definition of  $\mathcal{A}_\rho : D(\mathcal{A}_\rho) \subset \mathbf{H}_\rho \rightarrow \mathbf{H}_\rho$  in hand from (24)-(26), the resolvent relation (13) gives rise to the following fluid-structure PDE system:

$$\begin{cases} \omega_2 = \lambda\omega_1 - \omega_1^* & \text{in } \Omega \\ (\alpha^2 - \beta^2)\omega_1 + 2i\alpha\beta\omega_1 + P_\rho^{-1}\Delta^2\omega_1 - P_\rho^{-1}\pi_0|_\Omega = \omega_2^* + \lambda\omega_1^* & \text{in } \Omega \\ \omega_1|_{\partial\Omega} = \frac{\partial\omega_1}{\partial n}|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases} \quad (29)$$

$$\begin{cases} \lambda\mu - \Delta\mu + \nabla\pi_0 = \mu^* & \text{in } \mathcal{O} \\ \operatorname{div}(\mu) = 0 & \text{in } \mathcal{O} \\ \mu = 0 \text{ on } S; \quad \mu = [0, 0, \lambda\omega_1 - \omega_1^*] & \text{in } \Omega. \end{cases} \quad (30)$$

Here, the pressure term is given, via (24)-(26), as

$$\pi_0 = G_{\rho,1}(\omega_1) + G_{\rho,2}(\mu) \in H^1(\mathcal{O}), \text{ for } [\omega_1, \omega_2, \mu] \in D(\mathcal{A}_\rho), \quad (31)$$

where  $G_{\rho,1}$  and  $G_{\rho,2}$  are given by (22) and (23).

The proof of Theorem 3 will ultimately depend on the appropriate use of four basic relations:

(i) Taking the  $D(P_\rho^{\frac{1}{2}})$ -inner product of both sides of the structural PDE in (29) with  $\omega_1$ , integrating by parts and subsequently taking the real part of the result, we have

$$\alpha^2(1 - \tan^2 \vartheta) \left\| P_\rho^{\frac{1}{2}} \omega_1 \right\|_{L^2(\Omega)}^2 + \|\Delta\omega_1\|_{L^2(\Omega)}^2 = \operatorname{Re}(\pi_0|_\Omega, \omega_1)_\Omega + \operatorname{Re}(P_\rho[\omega_2^* + \lambda\omega_1^*], \omega_1)_\Omega, \quad (32)$$

(after also using implicitly the Criterion 1 above).

(ii) We take the  $\mathbf{L}^2(\mathcal{O})$ -inner product of both sides of the fluid PDE in (30) with  $\mu$ . After integrating by parts and then taking the respective imaginary and real parts of the resulting relation, we have,

$$\beta \|\mu\|_{\mathcal{O}}^2 = -\operatorname{Im}(\pi_0|_\Omega, \lambda\omega_1 - \omega_1^*)_\Omega + \operatorname{Im}(\mu^*, \mu)_{\mathcal{O}}; \quad (33)$$

$$\alpha \|\mu\|_{\mathcal{O}}^2 + \|\nabla\mu\|_{\mathcal{O}}^2 = -\operatorname{Re}(\pi_0|_\Omega, \lambda\omega_1 - \omega_1^*)_\Omega + \operatorname{Re}(\mu^*, \mu)_{\mathcal{O}}. \quad (34)$$

(iii) Lastly, we take the  $\mathbf{H}_\rho$ -inner product of both sides of the resolvent equation (13) with respect to solution variables  $[\omega_1, \omega_2, \mu]$ . This gives, upon integrating and taking the real part of the resulting relation:

$$\alpha \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2 = -\|\nabla\mu\|_{\mathcal{O}}^2 + \operatorname{Re} \left( \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix}, \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right)_{\mathbf{H}_\rho}. \quad (35)$$

In view of the right hand side of the relations (32) and (33)-(34), it is evidently necessary to scrutinize the “interface” term  $(\pi_0|_\Omega, \omega_1)_\Omega$ . Indeed, the estimation of this term will constitute the bulk of the effort in this work. By way of attaining a useful estimation, we will need to consider the explicit representation of the pressure term  $\pi_0$ , as given in (31). Via this expression we have then,

$$(\pi_0|_\Omega, \omega_1)_\Omega = (G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega + (G_{\rho,2}(\mu)|_\Omega, \omega_1)_\Omega. \quad (36)$$

We will proceed now to estimate each inner product on the right hand side of (36).

### 3.1 Analysis of the Term $(G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega}$

We recall from (23) that

$$G_{\rho,2}(\mu) = R_{\rho}(\Delta\mu^3|_{\Omega}) + \tilde{R}_p(\Delta\mu \cdot \nu|_S). \quad (37)$$

With the right hand side of (37) in mind we define the positive, self-adjoint operator  $B_{\rho} : D(B_{\rho}) \subset L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$  by

$$B_{\rho}f = -\Delta f \text{ in } \mathcal{O}; \quad D(B_{\rho}) = \left\{ f \in H^1(\mathcal{O}) : \Delta f \in L^2(\mathcal{O}) \text{ and } \begin{cases} \frac{\partial f}{\partial \nu} + P_{\rho}^{-1}f = 0 & \text{on } \Omega \\ \frac{\partial f}{\partial \nu} = 0 & \text{on } S \end{cases} \right\}. \quad (38)$$

Therewith one can readily compute the respective adjoints of  $R_{\rho} \in \mathcal{L}(H^{-\frac{1}{2}}(\Omega), H^1(\mathcal{O}))$ ,  $\tilde{R}_p \in \mathcal{L}(H^{-\frac{1}{2}}(S), H^1(\mathcal{O}))$ ,  $B_{\rho}R_{\rho} \in \mathcal{L}(H^{-\frac{1}{2}}(\Omega), [H^1(\mathcal{O})]')$  and  $B_{\rho}\tilde{R}_p \in \mathcal{L}(H^{-\frac{1}{2}}(S), [H^1(\mathcal{O})]')$ , as,

$$R_{\rho}^*f = B_{\rho}^{-1}f|_{\Omega} \text{ for all } f \in [H^1(\mathcal{O})]'; \quad (39)$$

$$\tilde{R}_p^*f = B_{\rho}^{-1}f|_S \text{ for all } f \in [H^1(\mathcal{O})]'; \quad (40)$$

$$R_{\rho}^*B_{\rho}f = f|_{\Omega} \text{ for all } f \in [H^1(\mathcal{O})]; \quad (41)$$

$$\tilde{R}_p^*B_{\rho}f = f|_S \text{ for all } f \in [H^1(\mathcal{O})]. \quad (42)$$

Indeed, to show (39): Given  $g \in H^{-\frac{1}{2}}(\Omega)$  and  $f \in [H^1(\mathcal{O})]'$ , we have from (38)

$$\begin{aligned} (R_{\rho}g, f)_{\mathcal{O}} &= (R_{\rho}g, (-\Delta)B_{\rho}^{-1}f)_{\mathcal{O}} \\ &= (\nabla R_{\rho}g, \nabla B_{\rho}^{-1}f)_{\mathcal{O}} - (R_{\rho}g, \frac{\partial}{\partial \nu}B_{\rho}^{-1}f)_{\Omega} + 0 \\ &= (-\Delta R_{\rho}g, B_{\rho}^{-1}f)_{\mathcal{O}} + (\frac{\partial}{\partial \nu}R_{\rho}g, B_{\rho}^{-1}f)_{\Omega} + (R_{\rho}g, P_{\rho}^{-1}B_{\rho}^{-1}f)_{\Omega} \\ &= (\frac{\partial}{\partial \nu}R_{\rho}g + P_{\rho}^{-1}R_{\rho}g, B_{\rho}^{-1}f)_{\Omega} \\ &= (g, B_{\rho}^{-1}f)_{\Omega}. \end{aligned}$$

The proofs of relations (40)-(42) are similar.

With the relations (39)-(42) in hand, we proceed: From (37) we have

$$\begin{aligned} (G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega} &= (R_{\rho}^*B_{\rho} [R_{\rho}(\Delta\mu^3|_{\Omega}) + \tilde{R}_p(\Delta\mu \cdot \nu|_S)], \omega_1)_{\Omega} \\ &= ([R_{\rho}(\Delta\mu^3|_{\Omega}) + \tilde{R}_p(\Delta\mu \cdot \nu|_S)], B_{\rho}R_{\rho}\omega_1)_{\mathcal{O}} \\ &= (\Delta\mu^3|_{\Omega}, R_{\rho}\omega_1|_{\Omega})_{\Omega} + (\Delta\mu \cdot \nu|_S, R_{\rho}\omega_1|_S)_S \\ &= (\Delta\mu \cdot \nu, R_{\rho}\omega_1)_{\partial\mathcal{O}} \end{aligned}$$

(and we are also using here the fact that  $\nu|_{\Omega} = [0, 0, 1]$ ). Invoking now Green's Formula – and simultaneously using the fact that fluid term  $\Delta\mu$  is divergence free – yields

$$(G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega} = (\Delta\mu, \nabla R_{\rho}\omega_1)_{\mathcal{O}}. \quad (43)$$

Following this relation up with Green's First Identity, we have then

$$(G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega} = -(\nabla\mu, \nabla(\nabla R_{\rho}\omega_1))_{\mathcal{O}} + \left\langle \frac{\partial\mu}{\partial\nu}, \nabla R_{\rho}\omega_1|_{\partial\mathcal{O}} \right\rangle_{\partial\mathcal{O}}. \quad (44)$$

### 3.1.1 Estimating the term $|(\nabla\mu, \nabla(\nabla R_\rho\omega_1))_{\mathcal{O}}|$

At this point, we consider the term  $R_\rho\omega_1$  – where map  $R_\rho$  is again given by (18) – as the solution of the following elliptic problem: For  $\rho \geq 0$ , we see from (18) that  $R_\rho\omega_1$  solves the elliptic problem

$$\begin{aligned}\Delta(R_\rho\omega_1) &= 0 \text{ in } \mathcal{O} \\ \frac{\partial(R_\rho\omega_1)}{\partial\nu} &= [\omega_1 - P_\rho^{-1}(R_\rho\omega_1|_\Omega)]_{ext} \text{ on } \partial\mathcal{O},\end{aligned}\tag{45}$$

where  $L^2(\partial\mathcal{O})$ -Neumann data is given by

$$[\omega_1 - P_\rho^{-1}(R_\rho\omega_1|_\Omega)]_{ext} \equiv \begin{cases} \omega_1 - P_\rho^{-1}(R_\rho\omega_1|_\Omega) & \text{on } \Omega \\ 0 & \text{on } S. \end{cases}$$

Then by the regularity result in [10], valid for Lipschitz domains, we have the estimate

$$\begin{aligned}\|R_\rho\omega_1\|_{H^{\frac{3}{2}}(\mathcal{O})} &\leq C \left\| [\omega_1 - P_\rho^{-1}(R_\rho\omega_1|_\Omega)]_{ext} \right\|_{L^2(\partial\mathcal{O})} \\ &\leq C \left( \|\omega_1\|_{L^2(\Omega)} + \|P_\rho^{-1}(R_\rho\omega_1|_\Omega)\|_{L^2(\Omega)} \right) \\ &\leq C \|\omega_1\|_{L^2(\Omega)},\end{aligned}\tag{46}$$

where in the second to last inequality we have used  $P_\rho^{-1} \in \mathcal{L}(L^2(\Omega), D(P_\rho))$ , as well as the boundedness of  $R_\rho \in \mathcal{L}(H^{-\frac{1}{2}}(\Omega), \mathbf{H}^1(\mathcal{O}))$ , which is noted in (20).

Using the estimate in (46), in tandem with interpolation, we have now – using implicitly  $H^{\frac{1}{2}}(\mathcal{O}) = H_0^{\frac{1}{2}}(\mathcal{O})$ ; see e.g., Theorem 3.40 (i) of [15] –

$$\begin{aligned}(\nabla\mu, \nabla(\nabla R_\rho\omega_1))_{\mathcal{O}} &= \langle \nabla\mu, \nabla(\nabla R_\rho\omega_1) \rangle_{H^{\frac{1}{2}}(\mathcal{O}) \times H^{-\frac{1}{2}}(\mathcal{O})} \\ &\leq C \|\nabla\mu\|_{H^{\frac{1}{2}}(\mathcal{O})} \|\nabla(\nabla R_\rho\omega_1)\|_{H^{-\frac{1}{2}}(\mathcal{O})} \\ &\leq C \|\mu\|_{H^{\frac{3}{2}}(\mathcal{O})} \|\omega_1\|_{L^2(\Omega)} \\ &\leq C \|\mu\|_{H^1(\mathcal{O})}^{\frac{1}{2}} \|\mu\|_{H^2(\mathcal{O})}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)} \\ &\leq C \|\mu\|_{H^1(\mathcal{O})}^{\frac{1}{2}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_\rho)}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)} \\ &= C \frac{|\alpha|^{\frac{1}{4}}}{|\alpha|^{\frac{1}{4}}} \|\mu\|_{H^1(\mathcal{O})}^{\frac{1}{2}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_\rho)}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)}.\end{aligned}$$

Estimating further the right hand side of this inequality, via Young's Inequality, now yields

$$\begin{aligned}|(\nabla\mu, \nabla(\nabla R_\rho\omega_1))_{\mathcal{O}}| &\leq \frac{\epsilon}{|\alpha|} \|\nabla\mu\|_{\mathcal{O}}^2 + C_\epsilon |\alpha|^{\frac{1}{3}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_\rho)}^{\frac{2}{3}} \|\omega_1\|_{L^2(\Omega)}^{\frac{4}{3}} \\ &= \frac{\epsilon}{|\alpha|} \|\nabla\mu\|_{\mathcal{O}}^2 + C_\epsilon |\alpha| (1 + \tan^2 \vartheta)^{\frac{1}{3}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_\rho}^{\frac{2}{3}} \\ &\quad \times \|\omega_1\|_{L^2(\Omega)}^{\frac{4}{3}},\end{aligned}\tag{47}$$

where in the last step, we have used the resolvent relation (13), and the assumption in Criterion 1 that frequency domain parameter  $\lambda$  lies along one of the two rays  $e^{\pm i\vartheta}$ , for fixed  $\vartheta \in (3\pi/4, \pi)$ . Estimating once



more, we have then

$$\begin{aligned} |(\nabla\mu, \nabla(\nabla R_\rho\omega_1))_{\mathcal{O}}| &\leq \frac{\epsilon}{|\alpha|} \|\nabla\mu\|_{\mathcal{O}}^2 + \frac{\epsilon}{2} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2 \\ &\quad + C_{\epsilon, \vartheta, \delta} |\alpha|^{\frac{3}{2}} \|\omega_1\|_{\Omega}^2, \end{aligned}$$

or for  $|\alpha| > 1$ ,

$$|(\nabla\mu, \nabla(\nabla R_\rho\omega_1))_{\mathcal{O}}| \leq \frac{\epsilon}{|\alpha|} \|\nabla\mu\|_{\mathcal{O}}^2 + \epsilon \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta, \delta} |\alpha|^{\frac{3}{2}} \|\omega_1\|_{D(P_\rho^{\frac{1}{2}})}^2 + C_{\epsilon, \delta} \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2. \quad (48)$$

### 3.1.2 Estimating the term $\left| \left\langle \frac{\partial\mu}{\partial\nu}, \nabla R_\rho\omega_1 \right\rangle_{\partial\mathcal{O}} \right|$ for $\rho = 0$ .

The second term on the right hand side of (44) is an even more delicate matter; the analysis here necessarily becomes a dichotomy with respect to  $\rho = 0$  and  $\rho > 0$ . In either case, we will need the following boundary trace inequalities (see e.g., Theorem 1.6.6 of [6], p. 37): Let  $D$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with Lipschitz boundary  $\partial D$ . Then there is a positive constant  $C^*$  such that

$$\|f\|_{\partial D} \leq C^* \|f\|_{L^2(D)}^{\frac{1}{2}} \|f\|_{H^1(D)}^{\frac{1}{2}} \text{ for every } f \in H^1(D). \quad (49)$$

$$\left\| \frac{\partial f}{\partial\nu} \right\|_{\partial D} \leq C^* \|f\|_{H^1(D)}^{\frac{1}{2}} \|f\|_{H^2(D)}^{\frac{1}{2}} \text{ for every } f \in H^2(D). \quad (50)$$

Note that the second inequality follows from (49), after using the fact that normal vector  $\nu \in \mathbf{L}^\infty(\partial D)$ , since  $\partial D$  is Lipschitz; see [16] (and so constant  $C^*$  depends upon  $\|\nu\|_{\mathbf{L}^\infty(\partial D)}$ .)

To start: We will have need here of the following positive definite, self-adjoint operator  $\mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ , defined by

$$\mathring{\mathbf{A}}\varpi = \Delta^2\varpi, \quad D(\mathring{\mathbf{A}}) = H^4(\Omega) \cap H_0^2(\Omega). \quad (51)$$

(Note that in the case that  $\Omega$  is polygonal – i.e., geometric condition (G.2) is in force – the angle condition assumed in (G.2) assures the smoothness of  $D(\mathring{\mathbf{A}})$  as given; see Theorem 2 of [5]). As such, this operator obeys the following “analyticity” estimate for all  $s > 0$ :

$$\left\| \mathring{\mathbf{A}}^\eta \mathcal{R}(-s; \mathring{\mathbf{A}}) \right\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{C}{(1+s)^{1-\eta}}, \text{ for all } \eta \in [0, 1] \quad (52)$$

(see e.g., the expression (5.15) in [12], p. 115). With this operator in hand, then in the present case  $\rho = 0$  the structural equation in (29) can be written as

$$\left[ \alpha^2(1 - \tan^2 \vartheta) + \mathring{\mathbf{A}} \right] \omega_1 = -2i\alpha\beta\omega_1 + \pi_0|_{\Omega} + \omega_2^* + \lambda\omega_1^*$$

(after also using Criterion 1). Applying thereto the operator  $\mathring{\mathbf{A}}^\eta \mathcal{R}(-\alpha^2(1 - \tan^2 \vartheta); \mathring{\mathbf{A}})$  gives then

$$\mathring{\mathbf{A}}^\eta \omega_1 = \mathring{\mathbf{A}}^\eta \mathcal{R}(-\alpha^2(1 - \tan^2 \vartheta); \mathring{\mathbf{A}}) [2i\alpha\beta\omega_1 - \pi_0|_{\Omega} - \omega_2^* - \lambda\omega_1^*].$$

Subsequently applying the estimate (52), we then have for  $0 \leq \eta \leq 1$  and  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned} \left\| \mathring{\mathbf{A}}^\eta \omega_1 \right\|_\Omega &\leq \frac{C}{(1 + \alpha^2(1 - \tan^2 \vartheta))^{1-\eta}} \left[ |\alpha\beta| \|\omega_1\|_{L^2(\Omega)} + \|\pi_0|_\Omega\|_{L^2(\Omega)} + \|\omega_2^* + \lambda\omega_1^*\|_{L^2(\Omega)} \right] \\ &\leq C_\vartheta |\alpha|^{2\eta} \|\omega_1\|_\Omega + \frac{C_\vartheta}{|\alpha|^{1-2\eta}} (\|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0} + \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}). \end{aligned} \quad (53)$$

In obtaining this estimate, we have used  $|\beta| = |\alpha| |\tan \vartheta|$ , the expression (31), and the resolvent equation (13).

With estimate (53) in hand, we now estimate the second term on the right hand side of (44): Reinvoking the estimate (46) for the solution of 45) (with therein  $\rho = 0$ ), in combination with the trace inequality (50), we have for  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned} \left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_0 \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| &\leq \left\| \frac{\partial \mu}{\partial \nu} \right\|_{\mathbf{L}^2(\partial \mathcal{O})} \|\nabla R_0 \omega_1|_{\partial \mathcal{O}}\|_{L^2(\partial \mathcal{O})} \\ &\leq C \|\nabla \mu\|_{\mathbf{L}^2(\mathcal{O})}^{\frac{1}{2}} \|\mu\|_{\mathbf{H}^2(\mathcal{O})}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)} \\ &\leq C \|\nabla \mu\|_{\mathbf{L}^2(\mathcal{O})}^{\frac{1}{2}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_0)}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)} \\ &= C_\vartheta \sqrt{|\alpha|} \|\nabla \mu\|_{\mathbf{L}^2(\mathcal{O})}^{\frac{1}{2}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_0}^{\frac{1}{2}} \|\omega_1\|_{L^2(\Omega)}, \end{aligned}$$

where again we have implicitly used the resolvent relation (13). Using now the characterization

$$H^{\frac{1}{2}-\delta}(\Omega) \approx D(\mathring{\mathbf{A}}^{\frac{1}{8}-\frac{\delta}{4}}),$$

which can be inferred from the definition of the domain in (51) and [9], we have upon applying (53) to the right hand side of (??),

$$\begin{aligned} \left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_0 \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| &\leq C_\vartheta \sqrt{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_0}^{\frac{1}{2}} \times \\ &\quad \left[ |\alpha|^{\frac{1}{4}-\frac{\delta}{2}} \|\omega_1\|_\Omega + \frac{1}{|\alpha|^{\frac{3}{4}+\frac{\delta}{2}}} (\|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0} + \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}) \right] \\ &= \frac{C_\vartheta}{|\alpha|^{\frac{1}{4}}} \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} |\alpha|^{1-\frac{\delta}{2}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_0}^{\frac{1}{2}} \times \\ &\quad \left[ \|\omega_1\|_\Omega + \frac{1}{|\alpha|} (\|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0} + \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}) \right]. \end{aligned}$$

This gives now, via Young's Inequality, for  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned}
\left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_0 \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| &\leq \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\epsilon, \vartheta} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} - \frac{1}{|\alpha| e^{\pm i \vartheta} \sqrt{1 + \tan^2 \vartheta}} \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_0}^{\frac{2}{3}} \times \\
&\quad |\alpha|^{\frac{4}{3} - \frac{2\delta}{3}} \left[ \|\omega_1\|_{\Omega} + \frac{1}{|\alpha|} (\|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0} + \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}) \right]^{\frac{4}{3}} \\
&\leq \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0}^2 \\
&\quad + C_{\epsilon, \vartheta} |\alpha|^{2-\delta} \|\omega_1\|_{\Omega}^2 + \frac{C_{\epsilon, \vartheta}}{|\alpha|^{\delta}} \left( \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0}^2 + \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}^2 \right). \quad (54)
\end{aligned}$$

### 3.1.3 Estimating the term $\left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_{\rho} \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right|$ for $\rho > 0$ .

Using again the estimates in (46) and (50), along with the Sobolev Trace Theorem, we have for  $0 < \delta < \frac{1}{2}$ ,

$$\begin{aligned}
\left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_{\rho} \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| &\leq \left\| \frac{\partial \mu}{\partial \nu} \right\|_{\partial \mathcal{O}} \|\nabla R_{\rho} \omega_1|_{\partial \mathcal{O}}\|_{\partial \mathcal{O}} \\
&\leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \|\mu\|_{\mathbf{H}^2(\mathcal{O})}^{\frac{1}{2}} \|R_{\rho} \omega_1\|_{H^{2-\delta}(\mathcal{O})} \\
&\leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \|\mu\|_{\mathbf{H}^2(\mathcal{O})}^{\frac{1}{2}} \|\omega_1\|_{H^{\frac{1}{2}-\delta}(\Omega)} \\
&\leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \|\mu\|_{\mathbf{H}^2(\mathcal{O})}^{\frac{1}{2}} \|\omega_1\|_{\Omega}^{\delta} \|\omega_1\|_{H^1(\Omega)}^{1-\delta}.
\end{aligned}$$

Combining this with the fluid boundary condition in (30) and the resolvent relation (13), we have then for  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned}
& \left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_\rho \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| \\
& \leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \|\mu\|_{\mathbf{H}^2(\mathcal{O})}^{\frac{1}{2}} \left\| \frac{1}{\lambda} (\mu^3|_{\Omega} + \omega_1^*) \right\|_{\Omega}^{\delta} \|\omega_1\|_{H^1(\Omega)}^{1-\delta} \\
& \leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_\rho)}^{\frac{1}{2}} \left\| \frac{1}{\lambda} (\mu^3|_{\Omega} + \omega_1^*) \right\|_{\Omega}^{\delta} \|\omega_1\|_{H^1(\Omega)}^{1-\delta} \\
& \leq C \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i \vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_\rho}^{\frac{1}{2}} \|\mu^3|_{\Omega} + \omega_1^*\|_{\Omega}^{\delta} \|\omega_1\|_{H^1(\Omega)}^{1-\delta} \\
& = C \frac{|\alpha|^{\frac{1}{4}}}{|\alpha|^{\frac{1}{4}}} \|\nabla \mu\|_{\mathcal{O}}^{\frac{1}{2}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i \vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_\rho}^{\frac{1}{2}} \|\mu^3|_{\Omega} + \omega_1^*\|_{\Omega}^{\delta} \|\omega_1\|_{H^1(\Omega)}^{1-\delta} \\
& \leq \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + C_\epsilon |\alpha|^{\frac{1}{3}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i \vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_\rho}^{\frac{2}{3}} \|\mu^3|_{\Omega} + \omega_1^*\|_{\Omega}^{\frac{4\delta}{3}} \|\omega_1\|_{H^1(\Omega)}^{\frac{4(1-\delta)}{3}} \\
& \leq \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_\epsilon |\alpha|^{\frac{1}{2}} \left( \frac{\sqrt{|\alpha|}}{\sqrt{|\alpha|}} \right)^{\frac{\delta}{2}} \|\mu^3|_{\Omega} + \omega_1^*\|_{\Omega}^{2\delta} \|\omega_1\|_{H^1(\Omega)}^{2(1-\delta)} \\
& \quad + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2 \\
& \leq \frac{\epsilon}{(C^*)^2 |\alpha|^{\frac{1}{2}}} \|\mu^3|_{\Omega} + \omega_1^*\|_{\Omega}^2 + C_\epsilon |\alpha|^{\frac{1+\delta}{2(1-\delta)}} \|\omega_1\|_{H^1(\Omega)}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 \\
& \quad + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2,
\end{aligned}$$

where  $C^*$  is the positive constant from the interpolation inequality (49). We have now, for sufficiently large  $|\alpha| > 1$

$$\begin{aligned}
& \left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_\rho \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| \\
& \leq \frac{\epsilon}{(C^*)^2 |\alpha|^{\frac{1}{2}}} \|\mu|_{\partial \mathcal{O}}\|_{\partial \mathcal{O}}^2 + C_\epsilon |\alpha|^{\frac{1+\delta}{2(1-\delta)}} \|\omega_1\|_{H^1(\Omega)}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2 \\
& \leq \frac{\epsilon}{|\alpha|^{\frac{1}{2}}} \|\mu\|_{\mathcal{O}} \|\nabla \mu\|_{\mathcal{O}} + C_\epsilon |\alpha|^{\frac{1+\delta}{2(1-\delta)}} \|\omega_1\|_{H^1(\Omega)}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2 \\
& \leq C_\epsilon |\alpha|^{\frac{1+\delta}{2(1-\delta)}} \|\omega_1\|_{H^1(\Omega)}^2 + \frac{2\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + 2\epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2. \tag{55}
\end{aligned}$$

Upon a rescaling of small parameter  $\epsilon > 0$ , we have then the estimate for  $\rho > 0$ ,

$$\begin{aligned}
& \left| \left\langle \frac{\partial \mu}{\partial \nu}, \nabla R_\rho \omega_1|_{\partial \mathcal{O}} \right\rangle_{\partial \mathcal{O}} \right| \\
& \leq C_\epsilon |\alpha|^{\frac{1+\delta}{2(1-\delta)}} \|\omega_1\|_{H^1(\Omega)}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_\epsilon \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2. \tag{56}
\end{aligned}$$

Combining now (44), (48), (54), and (56), and taking  $|\alpha| > 0$  sufficiently large, we have finally for all  $\rho \geq 0$ , and fixed  $0 < \delta < \frac{1}{2}$ ,

$$\begin{aligned} |(G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega}| &\leq C_{\epsilon,\vartheta} |\alpha|^{2-\delta} \|\omega_1\|_{D(P_{\rho}^{\frac{1}{2}})}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 \\ &\quad + \left( \epsilon + \frac{C_{\epsilon,\vartheta}}{|\alpha|^{\delta}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_{\rho}}^2 + C_{\epsilon} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_{\rho}}^2. \end{aligned}$$

Taking finally  $\delta \equiv \frac{1}{2} - \epsilon$ , we have then for  $|\alpha| > 0$  sufficiently large, and  $\rho \geq 0$ ,

$$\begin{aligned} |(G_{\rho,2}(\mu)|_{\Omega}, \omega_1)_{\Omega}| &\leq C_{\epsilon,\vartheta} |\alpha|^{\frac{3}{2}+\epsilon} \|\omega_1\|_{D(P_{\rho}^{\frac{1}{2}})}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 \\ &\quad + \left( \epsilon + \frac{C_{\epsilon,\vartheta}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_{\rho}}^2 + C_{\epsilon,\vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_{\rho}}^2. \end{aligned} \quad (57)$$

## 3.2 Analysis of the Term $|(G_{\rho,1}(\omega_1)|_{\Omega}, \omega_1)_{\Omega}|$

Recall that the image

$$G_{\rho,1}(\omega_1) = R_{\rho}(P_{\rho}^{-1} \Delta^2 \omega_1) \quad (58)$$

(see (22) and (36)).

As before, this work will entail a dichotomy between  $\rho = 0$  and  $\rho > 0$ .

### 3.2.1 Analysis of the term $|(G_{\rho,1}(\omega_1)|_{\Omega}, \omega_1)_{\Omega}|$ for $\rho = 0$

In this case, we have from (22) and the expressions in (39) and (41),

$$\begin{aligned} (G_{0,1}(\omega_1)|_{\Omega}, \omega_1)_{\Omega} &= (R_0 \Delta^2 \omega_1|_{\Omega}, \omega_1)_{\Omega} \\ &= (\Delta^2 \omega_1, [R_0 \omega_1]_{\Omega})_{\Omega}. \end{aligned} \quad (59)$$

An integration by parts to right hand side then gives

$$(G_{0,1}(\omega_1)|_{\Omega}, \omega_1)_{\Omega} = \left\langle \frac{\partial \Delta \omega_1}{\partial n}, [R_0 \omega_1]_{\Omega} \right\rangle_{\partial \Omega} - (\nabla \Delta \omega_1, \nabla [R_0 \omega_1]_{\Omega})_{\Omega}. \quad (60)$$

*To estimate the first term on the right hand side of (60):* Using the trace estimate (50), the regularity of the term  $R_0 \omega_1$  which is posted in (46) – with therein  $\delta \equiv \frac{1}{2}$  – and the Sobolev Trace Theorem, we have

$$\begin{aligned} \left| \left\langle \frac{\partial \Delta \omega_1}{\partial n}, [R_0 \omega_1]_{\Omega} \right\rangle_{\partial \Omega} \right| &\leq \left\| \frac{\partial \Delta \omega_1}{\partial n} \right\|_{\partial \Omega} \| [R_0 \omega_1]_{\Omega} \|_{\partial \Omega} \\ &\leq C \|\Delta \omega_1\|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta \omega_1\|_{H^2(\Omega)}^{\frac{1}{2}} \|R_0 \omega_1\|_{H^{\frac{3}{2}}(\mathcal{O})} \\ &\leq C \|\omega_1\|_{H^3(\Omega)}^{\frac{1}{2}} \|\omega_1\|_{H^4(\Omega)}^{\frac{1}{2}} \|\omega_1\|_{\Omega} \\ &\leq C \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{4}} \|\omega_1\|_{H^4(\Omega)}^{\frac{1}{4}} \|\omega_1\|_{H^4(\Omega)}^{\frac{1}{2}} \|\omega_1\|_{\Omega} \\ &\leq C \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{4}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_0)}^{\frac{3}{4}} \|\omega_1\|_{\Omega}. \end{aligned}$$

Using once more the resolvent relation (13), we have for  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned}
\left| \left\langle \frac{\partial \Delta \omega_1}{\partial n}, [R_\rho \omega_1]_\Omega \right\rangle_{\partial \mathcal{O}} \right| &\leq C_\vartheta |\alpha|^{\frac{3}{4}} \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{4}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_0}^{\frac{3}{4}} \|\omega_1\|_\Omega \\
&\leq \epsilon \|\omega_1\|_{H^2(\Omega)}^2 + C_{\epsilon, \vartheta} |\alpha|^{\frac{6}{7}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_0}^{\frac{6}{7}} \|\omega_1\|_\Omega^{\frac{8}{7}} \\
&\leq C_{\epsilon, \vartheta} |\alpha|^{\frac{3}{2}} \|\omega_1\|_\Omega^2 + 2\epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}^2, \tag{61}
\end{aligned}$$

after again using Young's Inequality.

*To estimate the second term on the right hand side of (60):* Using again the regularity of the term  $R_0 \omega_1$  which is posted in (46) – with therein  $\delta \equiv \frac{1}{2}$ , and the Sobolev Trace Theorem, we have

$$\begin{aligned}
|(\nabla \Delta \omega_1, \nabla [R_0 \omega_1]_\Omega)_\Omega| &\leq \|\nabla \Delta \omega_1\|_\Omega \|\nabla [R_0 \omega_1]_\Omega\|_\Omega \\
&\leq \|\nabla \Delta \omega_1\|_\Omega \|\nabla [R_0 \omega_1]_\Omega\|_{H^1(\Omega)} \\
&\leq C \|\omega_1\|_{H^3(\Omega)} \|R_0 \omega_1\|_{H^{\frac{3}{2}}(\mathcal{O})} \\
&\leq C \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{2}} \|\omega_1\|_{H^4(\Omega)}^{\frac{1}{2}} \|\omega_1\|_\Omega \\
&\leq C_\vartheta \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{2}} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_0)}^{\frac{1}{2}} \|\omega_1\|_\Omega \\
&\leq C_\vartheta \sqrt{|\alpha|} \|\omega_1\|_{H^2(\Omega)}^{\frac{1}{2}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_0}^{\frac{1}{2}} \|\omega_1\|_\Omega,
\end{aligned}$$

after again using the resolvent relation (13), and taking  $|\alpha| > 0$  sufficiently large. Proceeding via Young's Inequality, we have now

$$\begin{aligned}
|(\nabla \Delta \omega_1, \nabla [R_0 \omega_1]_\Omega)| &\leq \epsilon \|\omega_1\|_{H^2(\Omega)}^2 + C_{\epsilon, \vartheta} |\alpha|^{\frac{2}{3}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_0}^{\frac{2}{3}} \|\omega_1\|_\Omega^{\frac{4}{3}} \\
&\leq C_{\epsilon, \vartheta} |\alpha| \|\omega_1\|_\Omega^2 + 2\epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}^2. \tag{62}
\end{aligned}$$

Applying (61) and (62) to the right hand side of (60) (and rescaling parameter  $\epsilon > 0$ ) now gives

$$|(G_{0,1}(\omega_1)|_\Omega, \omega_1)_\Omega| \leq C_{\epsilon, \vartheta} |\alpha|^{\frac{3}{2}} \|\omega_1\|_\Omega^2 + 2\epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_0}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_0}^2. \tag{63}$$

### 3.2.2 Analysis of the term $|(G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega|$ for $\rho > 0$

Again from (22) and the expressions in (39) and (41), we have

$$\begin{aligned}
(G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega &= (R_\rho P_\rho^{-1} \Delta^2 \omega_1|_\Omega, \omega_1)_\Omega \\
&= (P_\rho^{-1} \Delta^2 \omega_1, [R_\rho \omega_1]_\Omega)_\Omega. \tag{64}
\end{aligned}$$

At this point we reinvoke the positive definite, self-adjoint operator  $\mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset L^2(\Omega) \rightarrow L^2(\Omega)$  in (51); with this operator in mind, we recall the following characterizations (see [9]):

$$D(\mathring{\mathbf{A}}^\eta) \approx \begin{cases} \{\varpi \in H^{4\eta}(\Omega) : \varpi|_{\partial\Omega} = 0\}, & \text{for } \frac{1}{8} < \eta < \frac{3}{8} \\ \{\varpi \in H^{4\eta}(\Omega) : \varpi|_{\partial\Omega} = \frac{\partial\varpi}{\partial n}|_{\partial\Omega} = 0\}, & \text{for } \frac{3}{8} < \eta \leq 1. \end{cases} \quad (65)$$

Proceeding from (64) we have then,

$$\begin{aligned} ((G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega) &= (P_\rho^{-1} \mathring{\mathbf{A}} \omega_1, [R_\rho \omega_1]_\Omega)_\Omega \\ &= (\mathring{\mathbf{A}}^{\frac{5}{8}+\epsilon} \omega_1, \mathring{\mathbf{A}}^{\frac{3}{8}-\epsilon} P_\rho^{-1} [R_\rho \omega_1]_\Omega)_\Omega. \end{aligned} \quad (66)$$

Using in part the fact that  $\mathring{\mathbf{A}}^{\frac{3}{8}-\epsilon} P_\rho^{-1} [R_\rho \omega_1]_\Omega$  is in  $L^2(\Omega)$  continuously – after using once more the estimate (46), with  $\delta \equiv \frac{1}{2}$  – a majorization of right hand side then gives

$$\begin{aligned} |((G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega)| &\leq \left\| \mathring{\mathbf{A}}^{\frac{5}{8}+\epsilon} \omega_1 \right\|_\Omega \left\| \mathring{\mathbf{A}}^{\frac{3}{8}-\epsilon} P_\rho^{-1} [R_\rho \omega_1]_\Omega \right\|_{\partial\Omega} \\ &\leq C \|\omega_1\|_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})}^{\frac{1}{2}-4\epsilon} \|\omega_1\|_{D(\mathring{\mathbf{A}}^{\frac{3}{4}})}^{\frac{1}{2}+4\epsilon} \|\omega_1\|_\Omega \\ &\leq C \|\omega_1\|_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})}^{\frac{1}{2}-4\epsilon} \|[\omega_1, \omega_2, \mu]\|_{D(\mathcal{A}_\rho)}^{\frac{1}{2}+4\epsilon} \|\omega_1\|_{H^1(\Omega)}, \end{aligned}$$

where in the last inequality we have recalled (25) and (65). Using once more the resolvent relation (13), we have for  $|\alpha| > 0$  sufficiently large,

$$\begin{aligned} &|((G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega)| \\ &\leq C_\vartheta |\alpha|^{\frac{1}{2}+4\epsilon} \|\omega_1\|_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})}^{\frac{1}{2}-4\epsilon} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_\rho}^{\frac{1}{2}+4\epsilon} \|\omega_1\|_{H^1(\Omega)} \\ &\leq \epsilon \|\omega_1\|_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})}^2 + C_{\epsilon, \vartheta} |\alpha|^{\frac{2+16\epsilon}{3+8\epsilon}} \left\| [\omega_1, \omega_2, \mu] - \frac{[\omega_1^*, \omega_2^*, \mu^*]}{|\alpha| e^{\pm i\vartheta} \sqrt{1 + \tan^2 \vartheta}} \right\|_{\mathbf{H}_\rho}^{\frac{2+16\epsilon}{3+8\epsilon}} \|\omega_1\|_{H^1(\Omega)}^{\frac{4}{3+8\epsilon}} \\ &\leq C_{\epsilon, \vartheta} |\alpha|^{1+8\epsilon} \|\omega_1\|_{H^1(\Omega)}^2 + 2\epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2, \end{aligned} \quad (67)$$

after using once more the characterization (65).

Combining (63) and (67), we have then for all  $\rho \geq 0$  and  $|\alpha| > 0$  sufficiently large,

$$|((G_{\rho,1}(\omega_1)|_\Omega, \omega_1)_\Omega)| \leq |(G_{0,1}(\omega_1)|_\Omega, \omega_1)_\Omega| \leq C_{\epsilon, \vartheta} |\alpha|^{\frac{3}{2}} \|\omega_1\|_{D(P_\rho^{\frac{1}{2}})}^2 + \epsilon \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2. \quad (68)$$

### 3.3 The Proof Proper of Theorem 3

Applying the estimates (57) and (68), to the right hand side of the expression (31), and using the resolvent relation  $\lambda \omega_1 = \omega_2 + \omega_1^*$  yield the following lemma:

**Lemma 7** For  $\rho \geq 0$ , the solution variables  $[\omega_1, \omega_2, \mu]$  of the resolvent equation obey the following estimate, for  $|\alpha| > 0$  sufficiently large:

$$\begin{aligned} \|(\pi_0|_\Omega, \omega_1)_\Omega\| &\leq C_{\epsilon, \vartheta} |\alpha|^{\frac{3}{2}+\epsilon} \|\omega_1\|_{D(P_\rho^{\frac{1}{2}})}^2 + \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 \\ &\quad + \left( \epsilon + \frac{C_{\epsilon, \vartheta}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \left( \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2 \right) \\ &\leq \frac{\epsilon}{|\alpha|} \|\nabla \mu\|_{\mathcal{O}}^2 + \left( \epsilon + \frac{C_{\epsilon, \vartheta}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2. \end{aligned} \quad (69)$$

In completing the proof of Theorem 3, we bear in mind that Criteria 1 and 2 are imposed upon complex parameter  $\lambda = \alpha + i\beta$ .

*Step 1.* We apply the estimate (69) to the right hand side of (33), so as to have

$$\begin{aligned} |\beta| \|\mu\|_{\mathcal{O}}^2 &= |-\text{Im}(\pi_0|_\Omega, \lambda\omega_1 - \omega_1^*)_\Omega + \text{Im}(\mu^*, \mu)_\mathcal{O}| \\ &\leq C_\vartheta |\alpha| |\text{Im}(\pi_0|_\Omega, \omega_1)_\Omega| + |\text{Im}(\pi_0|_\Omega, \omega_1^*)_\Omega| + |\text{Im}(\mu^*, \mu)_\mathcal{O}| \\ &\leq |\text{Im}(\pi_0|_\Omega, \omega_1^*)_\Omega| + \epsilon C_\vartheta^* \|\nabla \mu\|_{\mathcal{O}}^2 \\ &\quad + C_\vartheta^* |\alpha| \left( \epsilon + \frac{C_\epsilon}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_\rho}^2 + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2, \end{aligned} \quad (70)$$

where above, positive constant  $C_\vartheta^*$  is independent of parameter  $\epsilon > 0$ .

Now as for the first term on the right hand side of (70): Since the datum  $\omega_1^*$  satisfies the compatibility condition  $\int_\Omega \omega_1^* d\Omega = 0$ , then there is a function  $\varphi(\omega_1^*) \in \mathbf{H}^1(\mathcal{O})$  which solves

$$\begin{aligned} \text{div}(\varphi) &= 0 \quad \text{in } \mathcal{O}; \\ \varphi &= 0 \quad \text{in } S; \\ \varphi &= [0, 0, \omega_1^*] \quad \text{in } \Omega, \end{aligned} \quad (71)$$

with the estimate

$$\|\nabla \varphi\|_{\mathcal{O}} \leq C \|\omega_1^*\|_{H^{\frac{1}{2}}(\Omega)} \quad (72)$$

(see e.g., p. 9 of [8]). With this solution variable  $\varphi(\omega_1^*)$  in hand, and by virtue of the geometry in play, we then have

$$\begin{aligned} (\pi_0|_\Omega, \omega_1^*)_\Omega &= - \left( \frac{\partial \mu}{\partial \nu}, \varphi \right)_{\partial \mathcal{O}} + (\pi_0 \nu, \varphi)_{\partial \mathcal{O}} \\ &\quad - (\nabla \mu, \nabla \varphi)_\mathcal{O} - (\Delta \mu, \varphi)_\mathcal{O} + (\nabla \pi_0, \varphi)_{\partial \mathcal{O}} + 0 \\ &= - (\nabla \mu, \nabla \varphi)_\mathcal{O} - \lambda (\mu, \varphi)_\mathcal{O} + (\mu^*, \varphi)_\mathcal{O}, \end{aligned}$$

after using the fluid equation in (30). We have then upon majorizing, with the use of the estimate (72), and for large  $|\alpha| > 0$

$$|(\pi_0|_\Omega, \omega_1^*)_\Omega| \leq \epsilon \left( \|\nabla \mu\|_{\mathcal{O}}^2 + |\alpha| \|\mu\|_{\mathcal{O}}^2 \right) + |\alpha| C_{\epsilon, \vartheta} \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_\rho}^2. \quad (73)$$



Applying this estimate to the right hand side of (70) now yields (after a rescaling of  $\epsilon > 0$ )

$$\begin{aligned} |\beta| \|\mu\|_{\mathcal{O}}^2 &\leq \epsilon C_{\vartheta}^* \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\vartheta}^* |\alpha| \left( \epsilon + \frac{C_{\epsilon}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_{\rho}}^2 \\ &\quad + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_{\rho}}^2, \end{aligned} \quad (74)$$

where again, positive constant  $C_{\vartheta}^*$  is independent of parameter  $\epsilon > 0$ .

*Step 2:* We invoke the relation (34):

$$\|\nabla \mu\|_{\mathcal{O}}^2 = |\alpha| \|\mu\|_{\mathcal{O}}^2 - \operatorname{Re}(\pi_0|_{\Omega}, \lambda \omega_1 - \omega_1^*)_{\Omega} + \operatorname{Re}(\mu^*, \mu)_{\mathcal{O}}.$$

Applying the estimates (74), (69), and (73) to right hand side now gives

$$\begin{aligned} \|\nabla \mu\|_{\mathcal{O}}^2 &\leq \epsilon C_{\vartheta}^* \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\vartheta}^* |\alpha| \left( \epsilon + \frac{C_{\epsilon}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_{\rho}}^2 \\ &\quad + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_{\rho}}^2. \end{aligned} \quad (75)$$

*Step 3:* We apply the estimate (75) to the right hand side of (35). This gives for large  $|\alpha| > 0$

$$\begin{aligned} |\alpha| \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2 &= \left\| \nabla \mu\|_{\mathcal{O}}^2 + \operatorname{Re} \left( \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix}, \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right) \right\|_{\mathbf{H}_{\rho}} \\ &\leq \epsilon C_{\vartheta}^* \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\vartheta}^* |\alpha| \left( \epsilon + \frac{C_{\epsilon}}{|\alpha|^{\frac{1}{2}-\epsilon}} \right) \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_{\rho}}^2 \\ &\quad + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_{\rho}}^2. \end{aligned} \quad (76)$$

*Step 4:* Taking  $\epsilon > 0$  sufficiently small in (76) (with again positive constant  $C_{\vartheta}^*$  being independent of parameter  $\epsilon > 0$ ), we have

$$\begin{aligned} (1 - \epsilon C_{\vartheta}^*) |\alpha| \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2 \\ \leq \epsilon C_{\vartheta}^* \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\epsilon, \vartheta} \frac{|\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2 + C_{\epsilon, \vartheta} |\alpha| \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2, \end{aligned}$$

and so

$$\begin{aligned} |\alpha| \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2 \\ \leq \frac{\epsilon C_{\vartheta}^*}{1 - \epsilon C_{\vartheta}^*} \|\nabla \mu\|_{\mathcal{O}}^2 + C_{\epsilon, \vartheta} \frac{|\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2 + C_{\epsilon, \vartheta} |\alpha| \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_{\rho}}^2. \end{aligned} \quad (77)$$

*Step 5:* We return to the estimate (75). Applying (77) thereto gives for  $\epsilon > 0$  sufficiently small,

$$\begin{aligned} \|\nabla \mu\|_{\mathcal{O}}^2 &\leq \epsilon C_{\vartheta}^* \|\nabla \mu\|_{\mathcal{O}}^2 + \frac{C_{\epsilon, \vartheta} |\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_p}^2 \\ &\quad + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_p}^2, \end{aligned} \quad (78)$$

where again (relabelled) positive constant  $C_{\vartheta}^*$  does not depend upon small  $\epsilon > 0$ . Further specifying  $\epsilon > 0$  to be small enough, we have now

$$(1 - \epsilon C_{\vartheta}^*) \|\nabla \mu\|_{\mathcal{O}}^2 \leq \frac{C_{\epsilon, \vartheta} |\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_p}^2 + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_p}^2, \quad (79)$$

whence we obtain

$$\|\nabla \mu\|_{\mathcal{O}}^2 \leq \frac{C_{\epsilon, \vartheta} |\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \|[\omega_1, \omega_2, \mu]\|_{\mathbf{H}_p}^2 + C_{\epsilon, \vartheta} |\alpha| \|[\omega_1^*, \omega_2^*, \mu^*]\|_{\mathbf{H}_p}^2. \quad (80)$$

*Step 6:* We finish the proof by applying the estimate (80) to the right hand side of (77). This gives

$$|\alpha| \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_p}^2 \leq C_{\epsilon, \vartheta} \frac{|\alpha|}{|\alpha|^{\frac{1}{2}-\epsilon}} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_p}^2 + C_{\epsilon, \vartheta} |\alpha| \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_p}^2, \quad (81)$$

for fixed  $\epsilon > 0$ , small enough. Taking now  $|\alpha|$  so large that  $1 - \frac{C_{\epsilon, \vartheta}}{|\alpha|^{\frac{1}{2}-\epsilon}} > \frac{1}{2}$ ; i.e.,

$$|\alpha| > (2C_{\epsilon, \vartheta})^{\frac{2}{1-2\epsilon}},$$

we have finally

$$\frac{|\alpha|}{2} \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{\mathbf{H}_p}^2 \leq C_{\epsilon, \vartheta} |\alpha| \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{\mathbf{H}_p}^2, \quad (82)$$

which gives the uniform bound (14). This completes the proof of Theorem 3.

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